

Real analyticity of composition is shy

by

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Abstract. Dahmen and Schmeding have obtained the result that although the smooth Lie group G of real analytic diffeomorphisms $\mathbb{S}^{1\cdot} \rightarrow \mathbb{S}^{1\cdot}$ has a compatible analytic manifold structure, it does not make G a real analytic Lie group since the group multiplication is not real analytic. The authors considered this result “surprising” for the applied concept of infinite-dimensional real analyticity for maps $E \rightarrow F$, defined by the property that locally a holomorphic extension $E_{\mathbb{C}} \rightarrow F_{\mathbb{C}}$ exist. In this note we show that this type of real analyticity is quite rare for composition maps $f\varphi : x \mapsto \varphi \circ x$ when φ is real analytic. Specifically, we show that the smooth Fréchet space map $f\varphi : C(\mathbb{R}) \rightarrow C(\mathbb{R})$ for real analytic $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is real analytic in the above sense only if φ is the restriction to \mathbb{R} of some entire function $\mathbb{C} \rightarrow \mathbb{C}$. We also discuss the possibility of proving that the set of these “admissible” functions φ be “small” in the space $A(\mathbb{R})$ of real analytic functions either in the Baire categorical sense, or in the measure theoretic sense of shyness.

For maps $f : E \supseteq U \rightarrow F$, understood as triplets $\tilde{f} = (E, F, f)$ with $U = \text{dom } f$, where E and F are real Hausdorff locally convex spaces and f is a function between the underlying sets, there are several possibilities to reasonably define real analyticity of \tilde{f} . One is that of the “convenient calculus” developed in [15; p. 97ff.]. Another possibility is to represent f locally in some sense as a limit of partial sums of “power series”. A third possibility is to require locally existence of some “holomorphic” extension $E_{\text{cx}} \rightarrow F_{\text{cx}}$ between the complexifications, cf. [5; pp. 51–52]. This third approach further divides into several possibilities according to what kind of concept of holomorphy one chooses to use, cf. [1] and [16].

In this note, we shall use that third approach with holomorphy defined as meaning being C^∞ between complex Hausdorff locally convex spaces in the sense of [6] with topological vector spaces being interpreted as convergence vector spaces as explained there on page 236. We let $C_\Pi^\infty({}^{\text{tf}}\mathbb{C})$ denote the class of thus obtained holomorphic maps \tilde{f} . As explained in [6; Remarks 0.12, p. 241] for real scalars, noting that taking \mathbb{C} in place of \mathbb{R} in the required proofs does not change anything essential, our concept of holomorphy is precisely the same as that in [3; p. 23]. Hence also our associated real analyticity is precisely the same as there. See further [7; Theorem 3.8, pp. 14, 18] for the case where F is Mackey complete.

Let us say that \tilde{f} is *conveniently real analytic* in the case where $f : \text{dom } f \rightarrow F$ is real analytic in the sense of [15; Definition 10.3, p. 102].

For E in the class $\text{LCS}({}^{\text{tf}}\mathbb{R})$ of all real Hausdorff locally convex spaces, we let E_{cx} denote the *complexification* as explained in [3; A 2, p. 23]. Hence then $E_{\text{cx}} \in \text{LCS}({}^{\text{tf}}\mathbb{C})$ holds, and for the underlying sets we have $v_s(E_{\text{cx}}) = v_s E \times v_s E$.

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Here ${}^{\mathfrak{u}}\mathbb{R}$ and ${}^{\mathfrak{u}}\mathbb{C}$ are the real and complex topological fields whose underlying sets are \mathbb{R} and \mathbb{C} , respectively.

We will consider the maps $F\varphi = (E, E, f\varphi)$, where $E = C(\mathbb{R})$ is the real Fréchet space of continuous functions $\mathbb{R} \rightarrow \mathbb{R}$ with topology that of uniform convergence on bounded intervals, and $f\varphi = \langle \varphi \circ x : x \in v_s E \rangle$ with $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ real analytic. Thus $f\varphi$ is the function $v_s E \rightarrow v_s E$ defined by $x \mapsto \varphi \circ x$.

From our Theorem 1 below it follows that $F\varphi$ is smooth in all reasonable senses, and also conveniently real analytic. Contrary to this, by Theorem 2 it is real analytic in our sense only if φ has an entire extension.

Below, we let $f`x$ be the function value of f at x instead of the usual “ $f(x)$ ”. The zero vector of a topological (or any structured) vector space F is $\mathbf{0}_F$. In particular, for our fixed E above we have $\mathbf{0}_E = \mathbb{R} \times \{0\}$. We let \mathbf{U} be the class of all sets, and for functions f and g we have $[f, g]_f$ the function defined on $\text{dom } f \cap \text{dom } g$ by $x \mapsto (f`x, g`x)$. We put $\text{pr}_1 = \{(x, y, x) : x, y \in \mathbf{U}\}$, the global “first projection”, and $(z; x, y) = (z, (x, y))$. If $z = (x, y)$ is an ordered pair, then $x = \sigma_{\text{rd}} z$ and $y = \tau_{\text{rd}} z$. We further refer to [7; pp. 4–8], [8; pp. 4–9] and [9; p. 1] for a more extensive explanation of our notational system.

1 Theorem. *If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is smooth, then $F\varphi \in C_{\mathfrak{n}}^{\infty}({}^{\mathfrak{u}}\mathbb{R})$ holds. If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is real analytic, then $F\varphi$ is conveniently real analytic.*

Proof. The first assertion follows from [7; Theorem 3.6, p. 17] similarly as (a) in Remarks 3.7 there. For the second assertion, assuming the premise, by [15; Theorem 10.4, p. 102] for any continuous linear functional $\ell : E \rightarrow {}^{\mathfrak{u}}\mathbb{R}$ and for arbitrarily fixed $x, u \in v_s E$ it suffices to show that the real function $\chi : \mathbb{R} \rightarrow \mathbb{R}$ given by $t \mapsto \ell \circ f\varphi(x + tu)$ is real analytic on some open interval around zero. This in turn follows if we show that it has a holomorphic extension $\bar{\chi}$ around zero in some open set of the complex plane.

Now, by the Riesz representation [4; Theorem 7.4.1, p. 186] in conjunction with [10; Corollary, p. 262, Proposition 3.14.1, p. 266], there are a compact interval $I \subseteq \mathbb{R}$ and a bounded regular signed Borel measure μ on I with the property that $\ell`y = \int_I y \, d\mu$ holds for all $y \in v_s E$. Since φ has a holomorphic extension $\bar{\varphi}$ defined on some open set in the complex plane containing \mathbb{R} , and since $x`I$ and $u`I$ are compact, there is $\varepsilon \in \mathbb{R}^+$ such that for

$$\Omega = \{t + i\sigma : -1 < t < 1 \text{ and } -\varepsilon < \sigma < \varepsilon\}$$

we have $x`s + \zeta(u`s) \in \text{dom } \bar{\varphi}$ for all $s \in I$ and $\zeta \in \Omega$. Then defining $\bar{\chi} : \Omega \rightarrow \mathbb{C}$ by $\zeta \mapsto \int_I \bar{\varphi}(x`s + \zeta(u`s)) \, d\mu(s)$, we have $\bar{\chi}$ continuous with $\bar{\chi}|_{\mathbb{R}} \subseteq \chi$, and hence we are done if we show that $\bar{\chi}$ is holomorphic.

Letting Γ be the positively oriented boundary of an arbitrarily fixed closed triangle included in Ω , by Morera’s theorem it suffices to show that $\oint_{\Gamma} \bar{\chi} = 0$ holds. Now applying Fubini’s theorem separately to the positive and negative part of μ in its Jordan decomposition, we obtain

$$\begin{aligned} \oint_{\Gamma} \bar{\chi} &= \oint_{\Gamma} \int_I \bar{\varphi}(x`s + \zeta(u`s)) \, d\mu(s) \, d\zeta \\ &= \int_I \oint_{\Gamma} \bar{\varphi}(x`s + \zeta(u`s)) \, d\zeta \, d\mu(s) = 0. \end{aligned} \quad \square$$

2 Theorem. *If $\varphi : \mathbb{R} \rightarrow \mathbb{R}$ is real analytic with $F\varphi$ real analytic, then there is a holomorphic $\chi : \mathbb{C} \rightarrow \mathbb{C}$ with $\varphi = \chi|_{\mathbb{R}}$.*

Proof. Assuming the premise, with $E = C(\mathbb{R})$ as above, let $G = E_{\text{cx}}$ and $y = \mathbb{R} \times \{\varphi`0\}$. Now having $(\mathbf{0}_E, y) \in f\varphi$, some g exists such that $(\mathbf{0}_G; y, \mathbf{0}_E) \in g$

and $(G, G, g) \in C_{\mathbb{H}}^{\infty}({}^{\text{tf}}\mathbb{C})$ and $g \mid (\mathbf{U} \times \{\mathbf{0}_E\}) \subseteq [\text{f}\varphi \circ \text{pr}_1, \mathbf{U} \times \{\mathbf{0}_E\}]_{\text{f}}$ hold. Having $\mathbf{0}_G \in \text{dom } g \in \tau_{\text{rd}} G$, there is $n_0 \in \mathbb{Z}^+$ such that for

$$W = v_s G \cap \{z : \forall s; -n_0 \leq s \leq n_0 \Rightarrow |\sigma_{\text{rd}} z \backslash s| + |\tau_{\text{rd}} z \backslash s| < n_0^{-1}\}$$

we have $W \subseteq \text{dom } g$. With $m_0 = n_0 + 1$ and $J =]n_0, +\infty[$ taking

$$v = (\mathbb{R} \setminus J) \times \{0\} \cup \langle s - n_0 : s \in J \rangle \text{ and } w = (v, \mathbf{0}_E) \text{ and}$$

$\ell = \langle \sigma_{\text{rd}} z \backslash m_0 + i(\tau_{\text{rd}} z \backslash m_0) : z \in v_s G \rangle$, we obtain $\{\zeta w : \zeta \in \mathbb{C}\} \subseteq W$ with ℓ a continuous linear map $G \rightarrow {}^{\text{tf}}\mathbb{C}$. Hence for $\chi = \langle \ell \circ g \backslash (\zeta w) : \zeta \in \mathbb{C} \rangle$ we have $\chi : \mathbb{C} \rightarrow \mathbb{C}$ holomorphic. In addition, for $t \in \mathbb{R}$ we obtain

$$\begin{aligned} \chi \backslash t &= \ell \circ g \backslash (tv, \mathbf{0}_E) = \ell \circ [\text{f}\varphi \circ \text{pr}_1, \mathbf{U} \times \{\mathbf{0}_E\}]_{\text{f}} \backslash (tv, \mathbf{0}_E) \\ &= \ell \backslash (\text{f}\varphi \backslash (tv), \mathbf{0}_E) = \text{f}\varphi \backslash (tv) \backslash m_0 = \varphi \circ (tv) \backslash m_0 \\ &= \varphi \backslash (t(v \backslash m_0)) = \varphi \backslash (t1)) = \varphi \backslash t. \end{aligned}$$

□

The argument of the above proof of Theorem 2 *does not* work if instead we take the Fréchet space $E = C^{\infty}([0, 1])$. However, it is obvious that the same idea can be used to prove similar results for spaces $E = C^{\infty}(\Omega)$ when Ω is a nonempty open set in some “nonzero” Euclidean space.

3 Remark. Letting Ω be the set of all real analytic $x : \mathbb{R} \rightarrow \mathbb{R}$, and S its subset formed by the x possessing an entire extension, if one wants to consider whether S be “small” in Ω in some precise sense, one must put some structure on Ω . A standard procedure is to construct the locally convex space $A(\mathbb{R}) = (X, \mathcal{T}) = F$ with $v_s F = \Omega$ as follows. Let X be the (abstract) real vector space with underlying set Ω obtained by taking the “obvious” pointwise operations, and let \mathcal{T} be the strongest locally convex topology for X such that the identity is a continuous linear map $F U \rightarrow F$ for all $U \in \mathcal{U}$, when \mathcal{U} is the set of all open U in \mathbb{C} with $\mathbb{R} \subseteq U$, and $F U$ is the “obvious” Fréchet space of functions $x \in \Omega$ possessing a holomorphic extension $U \rightarrow \mathbb{C}$.

Letting \mathcal{B} be the Borel σ -algebra of the topological space (Ω, \mathcal{T}) , now to the above smallness one can give a precise mathematical content in one of the following two different ways:

(1) in the topological Baire categorical sense, here meaning that S is \mathcal{T} -meager in the sense that it can be expressed as a countable union of “rare” sets, i.e. those with closure having no interior points.

(2) in the measure theoretic sense, here meaning that S is F -shy in the sense that it is contained in a countable union of sets $B \in \mathcal{B}$ having the property that there is a finite-dimensional subspace M in X such that $B \cap \{x + v : v \in M\}$ for all $x \in B$ has Lebesgue measure zero in the “obvious” sense.

Using the result from [12] that $S \in \mathcal{B}$ holds, we can establish (2) quite easily. Namely, with any $u \in v_s F \setminus S$ taking $M = \{tu : t \in \mathbb{R}\}$ then as a singleton in a one-dimensional subspace $S \cap \{x + v : v \in M\} = \{x\}$ has measure zero. So by Theorem 2 the set $v_s A(\mathbb{R}) \cap \{\varphi : F\varphi \text{ is real analytic}\}$ is $A(\mathbb{R})$ -shy.

If instead of $A(\mathbb{R})$ we had for example $F = A([0, 1])$, which is a Silva space, an inductive limit of a sequence of Banach spaces with compact links, then we could easily establish (1) by noting that an elementary complex analysis argument using Cauchy’s formula shows that S is contained in a countable union of \mathcal{T} -compact, and hence rare sets. Since the sets $U \in \mathcal{U}$ in our actual situation are unbounded, this argument is not applicable. We can only show that S can be expressed as an *uncountable* union of $\tau_{\text{rd}} F U$ -compact, and hence \mathcal{T} -compact sets. So the question whether (1) holds in the above situation remains open.

4 Remark. When defining our concept of a set being “shy” in a topological vector space, we above deviated e.g. from the approaches in [2] and [11] since we wish to apply the concept to “highly nonmetrizable” spaces contrary to the cases *loc. cit.* where the underlying topology is assumed to be Polish, i.e. separable and completely metrizable. Specifically, we explicitly required the set to be contained in a countable union of “negligible” Borel sets since otherwise it might happen that a countable union of shy sets is not shy. In the restricted case of Polish topologies, one is able to give a quite nontrivial proof that a countable union of negligible Borel sets also is such. See [11; pp. 223–224] for the details.

Note further that for example for $F = {}^{\text{tr}}\mathbb{R}^{\mathbb{N}_0}_{\text{lcx}}$, the countable direct sum of the topological field ${}^{\text{tr}}\mathbb{R}$, we trivially have (1) that $v_s F$ is $\tau_{\text{rd}} F$ -meager, and (2) that $v_s F$ is F -shy. So an infinite-dimensional locally convex space can be both meager and shy “in itself” in the sense we defined in Remark 3 above. Another example is the Silva space $F = A([0, 1])$. In these cases $(v_s F, \tau_{\text{rd}} F)$ is not a Baire topological space, and being “shy” in F is not even defined in the sense of [11] since $\tau_{\text{rd}} F$ is not a metrizable topology.

If one wished to define shyness and its complement “prevalence” more carefully and generally, the generated σ algebra of \mathcal{A} being defined by

$$\sigma_{\text{Alg}} \mathcal{A} = \bigcap \{ \mathcal{B} : \mathcal{B} \text{ is a } \sigma\text{algebra and } \mathcal{A} \subseteq \mathcal{B} \},$$

one could put the following “semiformal”

5 Definitions. (1) Say that G is a *topologized group* iff there are g, Ω, \mathcal{T} such that $G = (g, \mathcal{T})$ and (Ω, \mathcal{T}) is a topological Hausdorff space and g is a group operation on Ω , and for all $x \in \Omega$ it holds that

$$\langle g^{\setminus}(x, y) : y \in \Omega \rangle \text{ and } \langle g^{\setminus}(y, x) : y \in \Omega \rangle \text{ are continuous } \mathcal{T} \rightarrow \mathcal{T}.$$

(2) Say that S is *shy* in G iff G is a topologized group and for all g, Ω, \mathcal{T} from $G = (g, \mathcal{T})$ and $\Omega = \bigcup \mathcal{T}$ it follows existence of a countable $\mathcal{A} \subseteq \sigma_{\text{Alg}} \mathcal{T}$ with $S \subseteq \bigcup \mathcal{A}$, and such that for every $A \in \mathcal{A}$ there are some \mathcal{T} -compact K and a probability measure μ with $\text{dom } \mu = \sigma_{\text{Alg}} \mathcal{T}$ and $(K, 1) \in \mu$, and such that $\langle g^{\setminus}(g^{\setminus}(x, z), y) : z \in A \rangle \in \mu^{-1} \setminus \{0\}$ holds for all $x, y \in \Omega$.

(3) Say that S is *prevalent* in G iff for all Ω from $\Omega = \bigcup \tau_{\text{rd}} G$ it follows that $S \subseteq \Omega$ and Ω is not shy in G and $\Omega \setminus S$ is shy in G .

(4) Say that F is *shy*^{LCS} *in itself* iff there is $\mathbf{K} \in \{{}^{\text{tr}}\mathbb{R}, {}^{\text{tr}}\mathbb{C}\}$ with $F \in \text{LCS}(\mathbf{K})$ and such that $v_s F$ is shy in $(\sigma_{\text{rd}}^2 F, \tau_{\text{rd}} F)$.

Note above that “ Ω is not shy in G ” is to be implicitly understood to mean that “it does not hold that Ω is shy in G ”. Further observe that 3(2) is a particular case of 5(2) since one can first restrict a Lebesgue measure to some “cube” of measure one, and then extend it by zero to all Borel sets.

As an application of our Definitions 5 above, we give the following

6 Proposition. *Every infinite-dimensional Silva space is shy^{LCS} in itself.*

Proof. With $\mathbf{K} \in \{{}^{\text{tr}}\mathbb{R}, {}^{\text{tr}}\mathbb{C}\}$ letting $F \in \text{LCS}(\mathbf{K})$ be an infinite-dimensional Silva space, there are $\mathbf{F} \in \text{BaS}(\mathbf{K})^{\mathbb{N}_0}$ and $\nu \in \mathbf{U}^{\mathbb{N}_0}$ with $F = \leq_{\text{LCS}}(\mathbf{K})$ -inf rng \mathbf{F} and such that for all $i \in \mathbb{N}_0$ we have $\mathbf{F}^{\setminus} i^+ \preceq \mathbf{F}^{\setminus} i$ with $\nu^{\setminus} i$ a compatible norm for $\mathbf{F}^{\setminus} i$ such that $(\nu^{\setminus} i)^{-1} \setminus [0, 1]$ is $\tau_{\text{rd}}(\mathbf{F}^{\setminus} i^+)$ -compact and $(\nu^{\setminus} i)^{-1} \setminus [0, 1] \subseteq (\nu^{\setminus} i^+)^{-1} \setminus [0, 1]$ holds. Since F is infinite-dimensional, we may also arrange matters so that $v_s(\mathbf{F}^{\setminus} i) \neq v_s(\mathbf{F}^{\setminus} i^+)$ holds.

Now putting $\mathbf{K} i j = (\nu^{\setminus} i)^{-1} \setminus [0, j^+ \cdot]$, we have $v_s F \subseteq \bigcup \{ \mathbf{K} i j : i, j \in \mathbb{N}_0 \}$, and for arbitrarily fixed $i, j \in \mathbb{N}_0$ it remains to construct some $\tau_{\text{rd}} F$ -compact

K and a probability measure μ with $\text{dom } \mu = \sigma_{\text{Alg}} \tau_{\text{rd}} F$ and $(K, 1) \in \mu$, and such that $\{(x+z)_{\text{svs } F} : z \in Kij\} \in \mu^{-i} \setminus \{0\}$ holds for all $x \in v_s F$. For this, we use a classical result of Alexandroff and Urysohn, see [14; Problem O(e), p. 166], guaranteeing existence of a surjection $\chi : \bigcup \mathcal{T} \rightarrow Kij$ which is continuous $\mathcal{T} \rightarrow \tau_{\text{rd}}(F \setminus i^+)$, when we take $\mathcal{T} = \mathcal{P}_s(2.)^{\mathbb{N}_0}_{\text{li}}$. On $\bigcup \mathcal{T}$ we then take the countable product measure μ_0 of $\{(\emptyset, 0), (1., \frac{1}{2}), (\{1.\}, \frac{1}{2}), (2., 1)\}$, and fixing any $x_0 \in v_s(F \setminus i^+) \setminus v_s(F \setminus i)$ we put $K = \{(x + s x_0)_{\text{svs } F} : x \in Kij \text{ and } 0 \leq s \leq 1\}$ and $\mu = \langle m A : A \in \sigma_{\text{Alg}} \tau_{\text{rd}} F \rangle$ where

$$m A = \int_{[0,1]} \mu_0(\chi^{-i} \setminus (Kij \cap \{x : (x + s x_0)_{\text{svs } F} \in A\})) \, d m_{\text{Leb}}(s).$$

It is a standard exercise in measure theory left to the reader to verify that these K and μ do the job we wished. \square

Note above that $E \preceq F$ means that $\text{id}_v F = \text{id}(v_s F)$ is a continuous linear map $F \rightarrow E$. In [7; p. 7] this was written (possibly) ambiguously “ $E \leq F$ ”.

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